On the forced motion due to heating of a deep rotating liquid in an annulus

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In the laboratory experiments by Fultz & Riehl (1957) and by Hide (1958) on heated rotating liquids contained in the annulus between two cylinders, it has been observed that a strongly marked jet stream appears on the free surface of the liquid under certain conditions of rotating and heating. This jet stream meanders around the annulus in a regular wave-like pattern alternately approaching the outer and inner cylindrical boundaries. The present paper puts forward an analytical theory for this jet, or Rossby regime, in the course of which exact solutions are presented of certain fundamental non-linear partial differential equations. The assumption is made that the flow is geostrophic at the first approximation and that the heat transfer across the stream lines of geostrophic flow (that is, the isobars) is due to molecular conduction. From a calculation of the heat flow it appears that this leads to values of the heat transfer which are too small, so that the ageostrophic terms must be of importance in the actual heat transfer; nevertheless, the exact solution obtained here probably reveals the mechanism of the change from one wave pattern to another and certainly provides an explanation for the observed upper limit to the number of waves in a given geometrical configuration, as discussed by Hide. It has been established that the mean zonal flow and mean zonal temperature field are dependent upon the amplitude function of a finite amplitude wave solution. In this exact solution it is found that the amplitude and phase functions of the wave patterns are themselves interdependent and that the shape of the wave depends on the quantity of heat and angular momentum being transferred. It is shown that the wave pattern consisting of an integral number m lobes or petals can exist only in a restricted range of the Rossby number S—this is well-known from the experimental work of Fultz and Hide.

Introduction

In the laboratory experiments by Fultz & Riehl (1957) and Hide (1958), liquid contained in the annulus between two concentric circular cylinders (radii b and $a\,(<b)$: see figure 1) is bounded below by a horizontal smooth surface and bounded above by a free surface. The cylinders are constrained to rotate steadily about their common axis, the outer cylinder being maintained at a temperature T_b and the inner at a temperature $T_a\,(<T_b)$. Under certain circumstances of rotation and heating, the forced flow of the liquid relative to the cylinders consists of a well-marked wave pattern in which the fluid motion is principally horizontal and in

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which a clearly defined jet stream can be observed on the free upper surface. The jet meanders regularly around the wave pattern, and the whole wave pattern plus the jet rotates relatively to the cylinders. A particular m-lobed wave pattern may be maintained indefinitely under constant conditions of rotation and temperature difference $T_b - T_a$, but can be changed by variation of either of these quantities. A discussion of the stability of the wave pattern has been made by the present author in an earlier paper (1956), and in this paper it was shown theoretically that the stability of the wave pattern is dependent principally upon one parameter known as the Rossby number S. In such stability analyses one is forced to assume that the wave amplitudes of the perturbation are infinitesimal compared, say, with the annulus radius $(b-a)$, and the results of such analyses will apply only to the initial stages of the wave growth under unstable conditions. The waves have an amplitude which is almost as large as the distance $(b-a)$, and this finiteness of amplitude can no longer be ignored in the investigation of the jet stream problem. This implies that the analytical investigation of the jet stream problem is necessarily one involving non-linear equations; until recently the only attempts which had been made to understand the jet stream structures mathematically were by numerical methods (Phillips 1956).

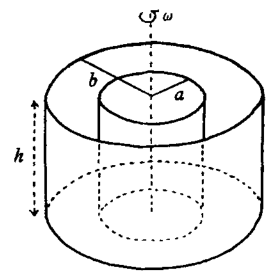

FIGURE 1. Definition sketch.

An analytical approach to the problem is possible, however, and the first step in this direction has been made by Miss Ruth Rogers (1959) who has investigated rectilinear jets using thermal boundary layer concepts in the heat transfer equation. The existence of such a rectilinear jet solution suggested to the present author that it might be profitable to investigate the corresponding jet problem in cylindrical co-ordinates and the present paper is a summary of the findings. It is interesting to note, however, that the boundary layer type of approximation made by Miss Rogers is not necessary in dealing with the present cylinder problem, and the solutions are correspondingly more valuable.

In the solution presented here it is found convenient to introduce amplitude and phase functions for the wave, and it then emerges that the zonal temperature field and the zonal flow can be expressed in terms of the amplitude and phase functions, which is something one would have expected from the many qualitative discussions of the corresponding meteorological problem (see, for instance, Lorenz 1957). It is found that the amplitude function and phase function are interdependent: one of these functions must be postulated before the complete solution can be obtained. Some guidance in postulating the nature of the phase function can be obtained from investigating the angular momentum and heat transfer associated with the wave. Particular examples illustrating different types of transfer have been included in this paper. One of the important results obtained is that the shape of the wave is dependent upon not only the amount of angular momentum transferred but also the amount of heat transferred; with an m-lobed wave pattern it is possible to transfer different amounts of heat and angular momentum within certain ranges. The m-lobed wave pattern is shown to exist only in certain ranges of the rotation and Rossby number, and when these ranges are combined with those obtained by the author in the earlier stability paper (1956) it is then possible to understand the complete stability diagram which has been obtained experimentally by Fultz (1956), and in particular to understand why there is a maximum number of waves possible in any given geometrical configuration. A result of secondary importance is that the steady solutions obtained here exist only if the ratio of the inner radius to the outer radius of the annulus is greater than a certain critical quantity. This implies that the steady wave regime cannot be maintained permanently in the open dishpan experiment.

A comparison of the theoretical absolute heat transport across either cylindrical boundary with that observed by Fultz indicates a variation with m in the correct sense, but the magnitude is considerably in error.

1. Statement of problem and method of solution

The dynamical equations of motion and the equation of heat transfer for a liquid contain terms which arise from the viscous stresses. One of the most striking features of the experiments conducted by Fultz & Riehl and by Hide is the negligible diffusion of the jet stream—a feature in which one might expect the viscous stresses to play a fundamental role. It is clear from order of magnitude arguments that the geostrophic approximation and the consequent 'thermal wind' together give results for the velocity field which are already good approximations (Davies 1953); and it would appear, although it is not possible to state this with precision, that there is probably a primary balance in the dynamical equations between the Coriolis acceleration terms and the pressure gradient (that is a geostrophic balance) and a separate but secondary balance between the viscous and inertia terms as in boundary-layer theory. Accordingly, in this non-viscous formulation of a theory of the experiments, molecular viscosity will be ignored completely.

In the absence of any heating the mean density of the liquid is taken to be ρ_0, and the small departure from this density due to a temperature increase τ above the mean is taken to be $-\alpha\tau$, where $\alpha = 2{\cdot}55\times10^{-4}\,\mathrm{g/cm^3}$ degree. Axes will be chosen which rotate steadily at an angular velocity ω about the fixed central z-axis, and the equations governing the motions will then be

$$\frac{du^*}{dt^*}-2\omega v^* = -\frac{1}{\rho_0}\frac{\partial p^*}{\partial x^*}, \tag{1.1}$$

$$\frac{dv^*}{dt^*}+2\omega u^* = -\frac{1}{\rho_0}\frac{\partial p^*}{\partial y^*}, \tag{1.2}$$

$$\rho_0\frac{dw^*}{dt^*} = -\frac{\partial p^*}{\partial z^*}+g\alpha\tau^*, \tag{1.3}$$

$$\frac{\partial u^*}{\partial x^*}+\frac{\partial v^*}{\partial y^*}+\frac{\partial w^*}{\partial z^*} = 0, \tag{1.4}$$

$$\frac{d\tau^*}{dt^*} = \kappa\nabla^2\tau^*, \tag{1.5}$$

where κ is the thermometric conductivity. Suppose that the typical temperature difference imposed externally in this problem is $(\Delta T)_H$, the suffix referring to a horizontal temperature contrast; then, if the liquid depth is h, we can introduce the non-dimensional symbols $x, y, \ldots, p$ defined by

$$\left.\begin{aligned} x^* = bx, \quad y^* = by, \quad z^* = hz; \\ u^* = 2\omega b S u, \quad v^* = 2\omega b S v, \quad w^* = 2\omega h S w; \\ t^* = t/2\omega S, \quad \tau^* = (\Delta T)_H \tau, \quad p^* = 4b^2\omega^2\rho_0 S p; \end{aligned}\right\} \tag{1.6}$$

where the Rossby number S is defined by

$$S = \frac{gh}{4b^2\omega^2}\frac{\alpha(\Delta T)_H}{\rho_0}. \tag{1.7}$$

With these transformations the equations (1.1) to (1.5) become

$$S\frac{du}{dt} - v = -\frac{\partial p}{\partial x}, \tag{1.8}$$

$$S\frac{dv}{dt} + u = -\frac{\partial p}{\partial y}, \tag{1.9}$$

$$\frac{h^2}{b^2}S\frac{dw}{dt} = -\frac{\partial p}{\partial z} + \tau, \tag{1.10}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{1.11}$$

$$S\frac{d\tau}{dt} = \frac{\kappa}{2\omega}\left\{\frac{1}{b^2}\nabla_H^2\tau + \frac{1}{h^2}\frac{\partial^2\tau}{\partial z^2}\right\}, \tag{1.12}$$

where

$$\nabla_H^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{1.13}$$

Now the Rossby number in jet flows is a small quantity of the order 0·1 to 0·01, and it is clear that the leading approximations to equations (1.8) to (1.12) will be as follows:

$$u_0 = -\frac{\partial p_0}{\partial y}, \quad v_0 = \frac{\partial p_0}{\partial x}, \quad w_0 = 0, \quad \tau_0 = \frac{\partial p_0}{\partial z}. \tag{1.14}$$

The first two equations represent the well-known geostrophic approximations for the horizontal motion, and combined with the fourth equation they lead to the equally well-known thermal wind equation of meteorology. The zero vertical velocity is a consequence of the vanishing of the horizontal divergence for the flow (u_0, v_0) coupled with (1.11). The heat transfer equation we treat in a more exact way, since if the conductivity is ignored it then follows that the isotherms and isobars everywhere coincide and there can be no heat transfer across the flow. We therefore retain the conductivity terms in (1.12) in order to ensure that there is a transfer of heat across the flow. On the left-hand side of (1.12), d/dt is an operator following the motion; but if we choose the angular velocity ω to be the angular velocity with which the wave system rotates relative to fixed axes in space, then the $\partial/\partial t$ term in d/dt can be omitted and the first approximation to (1.12) is

$$S\left\{u_0\frac{\partial\tau_0}{\partial x} + v_0\frac{\partial\tau_0}{\partial y}\right\} = \frac{\kappa}{2\omega}\left\{\frac{1}{b^2}\nabla_H^2\tau_0 + \frac{1}{h^2}\frac{\partial^2\tau_0}{\partial z^2}\right\}. \tag{1.15}$$

Equations (1.14) and (1.15) represent the equations of the first approximation, and the equations governing higher approximations will be developed later in this paper. When we substitute for u_0, v_0, τ_0 from (1.14) in (1.15), the resulting equation is a non-linear partial differential equation for p_0, namely

$$S\left\{\frac{\partial p_0}{\partial x}\frac{\partial \tau_0}{\partial y}-\frac{\partial p_0}{\partial y}\frac{\partial \tau_0}{\partial x}\right\}=\frac{\kappa}{2\omega}\left\{\frac{1}{b^2}\nabla^2_H\tau_0+\frac{1}{h_{\frac{3}{4}}^2}\frac{\partial^2\tau_0}{\partial z^2}\right\}\quad\left(\tau_0=\frac{\partial p_0}{\partial z}\right). \tag{1.16}$$

This is the equation derived originally by Miss Ruth Rogers (1959) in the investigation of the rectilinear jet.

One of the first points of interest concerning (1.16) is that to any solution found for τ_0 may be added an arbitrary linear function of z. Detailed measurements of the three-dimensional temperature field in the annulus experiment have been given by Fultz & Riehl (1957), and it is clear from these results that the mean vertical temperature field is approximately a linear function of z (except for an evaporation layer near the free surface). When this linear variation of temperature is subtracted from τ_0, the resulting temperature variation in the vertical is slow; hence if we introduce a new temperature function T_0 defined by

$$\tau_0 = cz+d+T_0, \tag{1.17}$$

and a corresponding new pressure function P_0 which differs from p_0 by a quadratic function of z, then T_0 and P_0 will have a slow variation in the vertical, and it will be sufficiently accurate, in the case of a liquid whose depth is large compared with the horizontal dimension, to simplify (1.16) to the form

$$\frac{\partial P_0}{\partial x}\frac{\partial T_0}{\partial y}-\frac{\partial P_0}{\partial y}\frac{\partial T_0}{\partial x}=\epsilon\left(\frac{\partial^2 T_0}{\partial x^2}+\frac{\partial^2 T_0}{\partial y^2}\right)\quad\left(T_0=\frac{\partial P_0}{\partial z}\right), \tag{1.18}$$

where

$$\epsilon = \kappa/2\omega b^2 S. \tag{1.19}$$

The non-dimensional constant ϵ is nothing more than a scaling constant, for if we write $P_0 = \epsilon P_0'$, $T_0 = \epsilon T_0'$, this constant cancels out in (1.17). It is a little more convenient, however, to write $P_0 = \epsilon P_0'/m$, $T_0 = \epsilon T_0'/m$, where m is the wave number (see (1.22)), and we shall consider (1.18) in the form

$$\frac{\partial P_0}{\partial x}\frac{\partial T_0}{\partial y}-\frac{\partial P_0}{\partial y}\frac{\partial T_0}{\partial x}=m\left(\frac{\partial^2 T_0}{\partial x^2}+\frac{\partial^2 T_0}{\partial y^2}\right)\quad\left(T_0=\frac{\partial P_0}{\partial z}\right). \tag{1.20}$$

(The dashes have not been retained in the dependent variables of this equation.) It may be noted that, using typical experimental values for all the constants, the value of the constant ϵ is about 10^{-3}.

We shall consider equation (1.20) in terms of polar co-ordinates (r, θ), where $x = r\cos\theta$, $y = r\sin\theta$. In this case (1.20) becomes

$$\frac{\partial P_0}{\partial r}\frac{\partial T_0}{\partial \theta}-\frac{\partial P_0}{\partial \theta}\frac{\partial T_0}{\partial r}=m\left\{r\frac{\partial^2 T_0}{\partial r^2}+\frac{\partial T_0}{\partial r}+\frac{1}{r}\frac{\partial^2 T_0}{\partial \theta^2}\right\}. \tag{1.21}$$

It may be observed that equation (1.20) is invariant with respect to a complex transformation

$$a+ib = f(x+iy),$$

since

$$\frac{\partial P_0}{\partial x}\frac{\partial T_0}{\partial y}-\frac{\partial P_0}{\partial y}\frac{\partial T_0}{\partial x}=\left|\frac{d(a+ib)}{d(x+iy)}\right|^2\left\{\frac{\partial P_0}{\partial a}\frac{\partial T_0}{\partial b}-\frac{\partial P_0}{\partial b}\frac{\partial T_0}{\partial a}\right\}$$

and

$$\frac{\partial^2 T_0}{\partial x^2}+\frac{\partial^2 T_0}{\partial y^2}=\left|\frac{d(a+ib)}{d(x+iy)}\right|^2\left\{\frac{\partial^2 T_0}{\partial a^2}+\frac{\partial^2 T_0}{\partial b^2}\right\}.$$

However, no use has been made of this interesting feature in this paper, although it can be used in extending the solutions obtained by Miss Rogers.

One of the most striking features of the experimental jet flows is the existence of a periodic structure in the θ-direction in which there is one regular well-marked stable wave pattern with a particular wave number m. This suggests that we investigate a solution of (1.21) of the form

$$P_0 = f + g\sin m\theta + h\cos m\theta, \tag{1.22}$$

$$T_0 = F + G\sin m\theta + H\cos m\theta, \tag{1.23}$$

where m is an integer and where $f, g, \ldots, H$ are functions of r, z only, although the z-variation of each of these functions is slow. When we substitute (1.22) and (1.23) in the left-hand side of (1.21), this side of the equation becomes

$$m\left\{\frac{1}{2}\left(G\frac{\partial h}{\partial r} - g\frac{\partial H}{\partial r} - H\frac{\partial g}{\partial r} + h\frac{\partial G}{\partial r}\right) + \cos m\theta\left(G\frac{\partial f}{\partial r} - g\frac{\partial F}{\partial r}\right)\right.$$
$$+ \sin m\theta\left(h\frac{\partial F}{\partial r} - H\frac{\partial f}{\partial r}\right) + \tfrac{1}{2}\cos 2m\theta\left(G\frac{\partial h}{\partial r} - h\frac{\partial G}{\partial r} - g\frac{\partial H}{\partial r} + H\frac{\partial g}{\partial r}\right)$$
$$\left. + \tfrac{1}{2}\sin 2m\theta\left(G\frac{\partial g}{\partial r} - g\frac{\partial G}{\partial r} + h\frac{\partial H}{\partial r} - H\frac{\partial h}{\partial r}\right)\right\}. \tag{1.24}$$

When we substitute (1.23) in the right-hand side of (1.21), there will be no terms in $\cos 2m\theta$ or $\sin 2m\theta$; hence we must have

$$G\frac{\partial h}{\partial r} - h\frac{\partial G}{\partial r} - g\frac{\partial H}{\partial r} + H\frac{\partial g}{\partial r} = 0, \tag{1.25}$$

$$G\frac{\partial g}{\partial r} - g\frac{\partial G}{\partial r} + h\frac{\partial H}{\partial r} - H\frac{\partial h}{\partial r} = 0. \tag{1.26}$$

Comparing the remaining terms on the right-hand side with the corresponding ones in (1.24), we obtain the following three equations:

$$\frac{1}{2}\left(G\frac{\partial h}{\partial r} - g\frac{\partial H}{\partial r} - H\frac{\partial g}{\partial r} + h\frac{\partial G}{\partial r}\right) = r\frac{\partial^2 F}{\partial r^2} + \frac{\partial F}{\partial r}, \tag{1.27}$$

$$G\frac{\partial f}{\partial r} - g\frac{\partial F}{\partial r} = r\frac{\partial^2 H}{\partial r^2} + \frac{\partial H}{\partial r} - \frac{m^2}{r^2}H, \tag{1.28}$$

$$h\frac{\partial F}{\partial r} - H\frac{\partial f}{\partial r} = r\frac{\partial^2 G}{\partial r^2} + \frac{\partial G}{\partial r} - \frac{m^2}{r^2}G. \tag{1.29}$$

In addition to the five equations (1.25) to (1.29), it follows also from $T_0 = \partial P_0/\partial z$ that

$$F = \frac{\partial f}{\partial z}, \tag{1.30}$$

$$G = \frac{\partial g}{\partial z}, \tag{1.31}$$

$$H = \frac{\partial h}{\partial z}. \tag{1.32}$$

If we had retained the term $\partial^2\tau_0/\partial z^2$, the only difference to the above equations would be the addition of terms $\partial^2 F/\partial z^2$, $\partial^2 H/\partial z^2$ and $\partial^2 G/\partial z^2$ on the right-hand sides of (1.27), (1.28) and (1.29) respectively. The equations (1.25) to (1.32) are still a non-linear set, and the principal aim now will be to obtain solutions which satisfy prescribed boundary conditions at the cylindrical boundaries.

Consider first of all equations (1.25) and (1.26). These are linear equations in h and g, and if, for example, g is eliminated between these two equations a second-order equation in r will be obtained for h. Hence, the general solution for h will contain two arbitrary functions of z. It is not necessary to perform this elimination, for it may be seen that the solutions for g and h will be

$$g = A(z)H + B(z)G, \tag{1.33}$$

$$h = -A(z)G + B(z)H, \tag{1.34}$$

where A and B are arbitrary functions of z. Using (1.31) and (1.32) it now follows that G and H must satisfy the equations

$$G = \frac{\partial}{\partial z}\{A(z)H + B(z)G\}, \tag{1.35}$$

$$H = \frac{\partial}{\partial z}\{-A(z)G + B(z)H\}, \tag{1.36}$$

and it is clear that the ultimate solutions for G and H will depend upon the nature of the variation of the functions $A(z)$ and $B(z)$.

Consider now equation (1.27) which we can write in the form

$$r\frac{\partial^2 F}{\partial r^2} + \frac{\partial F}{\partial r} = \frac{1}{2}\frac{\partial}{\partial r}\{Gh - gH\};$$

using (1.33) and (1.34) this becomes

$$r\frac{\partial^2 F}{\partial r^2} + \frac{\partial F}{\partial r} = -\tfrac{1}{2}A(z)\frac{\partial}{\partial r}(G^2 + H^2),$$

which has the first integral

$$r\frac{\partial F}{\partial r} = -\tfrac{1}{2}A(z)(G^2 + H^2) + \chi(z), \tag{1.37}$$

where χ is a function of z only. It is convenient at this stage to introduce local amplitude and local phase functions Φ and Ψ respectively, in place of H and G, and we shall define these functions as follows:

$$G = \Phi \sin\Psi, \quad H = \Phi\cos\Psi. \tag{1.38}$$

In terms of these new functions, (1.37) becomes

$$r\frac{\partial F}{\partial r} = -\tfrac{1}{2}A(z)\Phi^2 + \chi(z); \tag{1.39}$$

and if for convenience we use R in place of r as independent variable, where

$$R = \log_e r, \quad r = e^R, \quad r\frac{\partial}{\partial r} = \frac{\partial}{\partial R}, \tag{1.40}$$

then (1.39) may be written in the form

$$\frac{\partial F}{\partial R} = -\tfrac{1}{2}A(z)\,\Phi^2 + \chi(z). \tag{1.41}$$

In terms of R, equations (1.28) and (1.29) may be written as follows:

$$G\frac{\partial f}{\partial R} - g\frac{\partial F}{\partial R} = \frac{\partial^2 H}{\partial R^2} - m^2 H, \tag{1.42}$$

$$h\frac{\partial F}{\partial F} - H\frac{\partial f}{\partial R} = \frac{\partial^2 G}{\partial R^2} - m^2 G. \tag{1.43}$$

If we multiply (1.42) by H, (1.43) by G and add, we obtain

$$(hG - gH)\frac{\partial F}{\partial R} = H\left(\frac{\partial^2 H}{\partial R^2} - m^2 H\right) + G\left(\frac{\partial^2 G}{\partial R^2} - m^2 G\right).$$

It is easily shown from (1.38) that

$$H\frac{\partial^2 H}{\partial R^2} + G\frac{\partial^2 G}{\partial R^2} = \Phi\frac{\partial^2 \Phi}{\partial R^2} - \Phi^2\left(\frac{\partial \Psi}{\partial R}\right)^2,$$

and thus when we make use of (1.33) and (1.34) we obtain

$$-A\Phi^2\frac{\partial F}{\partial R} = \Phi\frac{\partial^2 \Phi}{\partial R^2} - \Phi^2\left(\frac{\partial \Psi}{\partial R}\right)^2 - m^2\Phi^2. \tag{1.44}$$

Comparing (1.41) and (1.44), it now follows that Φ and Ψ are related by the equation

$$\Phi\frac{\partial^2 \Phi}{\partial R^2} - \Phi^2\left(\frac{\partial \Psi}{\partial R}\right)^2 - m^2\Phi^2 = -A\Phi^2\{-\tfrac{1}{2}A\Phi^2 + \chi\}. \tag{1.45}$$

Using (1.42) and (1.43) once more, multiplying the former by h, the latter by g and adding, we obtain

$$(hG - gH)\frac{\partial f}{\partial R} = h\left(\frac{\partial^2 H}{\partial R^2} - m^2 H\right) + g\left(\frac{\partial^2 G}{\partial R^2} - m^2 G\right).$$

Substituting in this equation for h and g from (1.33) and (1.34), we obtain

$$\begin{aligned}-A\Phi^2\frac{\partial f}{\partial R} &= (BH - AG)\left(\frac{\partial^2 H}{\partial R^2} - m^2 H\right) + (AH + BG)\left(\frac{\partial^2 G}{\partial R^2} - m^2 G\right)\\ &= B\left\{H\frac{\partial^2 H}{\partial R^2} + G\frac{\partial^2 G}{\partial R^2} - m^2(H^2 + G^2)\right\} - A\left\{G\frac{\partial^2 H}{\partial R^2} - H\frac{\partial^2 G}{\partial R^2}\right\};\end{aligned}$$

and from (1.38) it is easily shown that

$$H\frac{\partial^2 G}{\partial R^2} - G\frac{\partial^2 H}{\partial R^2} = \Phi^2\frac{\partial^2 \Psi}{\partial R^2} + 2\Phi\frac{\partial \Phi}{\partial R}\frac{\partial \Psi}{\partial R}.$$

Hence the equation for $\partial f/\partial R$ becomes

$$-A\Phi^2\frac{\partial f}{\partial R} = B\left\{\Phi\frac{\partial^2 \Phi}{\partial R^2} - \Phi^2\left(\frac{\partial \Psi}{\partial R}\right)^2 - m^2\Phi^2\right\} + A\left\{\Phi^2\frac{\partial^2 \Psi}{\partial R^2} + 2\Phi\frac{\partial \Phi}{\partial R}\frac{\partial \Psi}{\partial R}\right\}. \tag{1.46}$$

We may note at this stage that (1.41) or (1.44) is an equation which defines $\partial F/\partial R$ in terms of Φ and Ψ, and that (1.46) defines $\partial f/\partial R$ in terms of Φ and Ψ. Bearing in mind equation (1.30), which we can write alternatively in the form

$$\frac{\partial F}{\partial R} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial R}\right), \tag{1.47}$$

it follows that when we substitute for $\partial F/\partial R$ and $\partial f/\partial R$ we obtain a second-order partial differential relation between Φ and Ψ which, coupled with (1.45), gives a complete solution of the problem. This second relation which follows from (1.47) is

$$-\tfrac{1}{2}A\Phi^2+\chi = \frac{\partial}{\partial z}\left\{B(-\tfrac{1}{2}A\Phi^2+\chi)-\left(\frac{\partial^2\Psi}{\partial R^2}+\frac{2}{\Phi}\frac{\partial\Phi}{\partial R}\frac{\partial\Psi}{\partial R}\right)\right\}. \tag{1.48}$$

In order to exploit equations (1.45) and (1.48) it is necessary to be able to separate the variables R and z. Without going into detail, it is sufficient to state that this is possible if, and only if,

$$\Phi = \frac{1}{A(z)}\phi(R), \tag{1.49}$$

$$\chi(z) = \frac{j}{A(z)}, \tag{1.50}$$

and $\partial\Psi/\partial R$ is independent of z, where $\phi(R)$ is a function of R only and j is a constant. If one uses the above formulae, it follows that $\partial F/\partial R$ is then given by

$$\frac{\partial F}{\partial R} = \frac{1}{A(z)}\{-\tfrac{1}{2}\phi^2(R)+j\}; \tag{1.51}$$

$\phi(R)$ satisfies the equation

$$\frac{d^2\phi}{dR^2}-\phi\left(\frac{d\Psi}{dR}\right)^2-m^2\phi = -\phi(-\tfrac{1}{2}\phi^2+j); \tag{1.52}$$

$\partial f/\partial R$ is given by

$$\begin{aligned}-\phi^2\frac{\partial f}{\partial R} &= \frac{B}{A}\phi\left\{\frac{d^2\phi}{dR^2}-\phi\left(\frac{d\Psi}{dR}\right)^2-m^2\phi\right\}+\frac{d}{dR}\left\{\phi^2\frac{d\Psi}{dR}\right\}\\ &= -\frac{B}{A}\phi^2(j-\tfrac{1}{2}\phi^2)+\frac{d}{dR}\left\{\phi^2\frac{d\Psi}{dR}\right\};\end{aligned} \tag{1.53}$$

and finally, in order to satisfy (1.48) or (1.47), we must have

$$\frac{d}{dz}\left(\frac{B}{A}\right) = \frac{1}{A}. \tag{1.54}$$

Although the method of separating the variables is not along the lines of the well-known method used for linear partial differential equations, it is convenient to regard j as the separation constant.

Before proceeding further it is useful to express the temperature and pressure in terms of the amplitude and phase functions. If we substitute for G and H from (1.38) in (1.23), we obtain

$$T_0 = F+\Phi\cos(m\theta-\Psi), \tag{1.55}$$

hence

$$T_0 = F+\frac{1}{A}\phi(R)\cos(m\theta-\Psi). \tag{1.56}$$

Similarly, if we substitute for g and h from (1.33), (1.34) in (1.22), we obtain

$$P_0 = f + A(H \sin m\theta - G \cos m\theta) + B(G \sin m\theta + H \cos m\theta),$$

hence, using (1.38),

$$P_0 = f + A\Phi \sin(m\theta - \Upsilon) + B\Phi \cos(m\theta - \Upsilon). \tag{1.57}$$

Finally, making use of (1.49), we have

$$P_0 = f + \phi(R) \sin(m\theta - \Upsilon) + \frac{B}{A}\phi(R) \cos(m\theta - \Upsilon). \tag{1.58}$$

It is clear that $\phi(R)$ can be determined from (1.52) when $d\Upsilon/dR$ is known, and that B can be determined from (1.54) when A is known; hence, the knowledge of $d\Upsilon/dR$ as a function of R and of A as a function of z will be sufficient (together with the boundary conditions) to determine ϕ, B, F and f. The unknown constant j will, we shall find, be determined as an eigen-value similar to linear theory. It is important to point out that $\partial\Upsilon(R, z)/\partial R$ is a function of R only and the function Υ is necessarily a function of R only, as may be verified from (1.35) or (1.36). This means that, with the present solution of the problem, that part of the temperature field which depends upon θ does not show variation in the vertical. This feature of the temperature field (not of the pressure field it may be noted) is contrary to the experiment where a phase change in the vertical is present in the temperature structure. This phase change will be discussed in greater detail in a future paper. To conclude this section we may note, from (1.51) and (1.53), that the fields of mean temperature F and mean pressure f satisfy the equation

$$\frac{\partial}{\partial R}\{f - BF\} = -\frac{1}{\phi^2}\frac{d}{dR}\left(\phi^2 \frac{d\Upsilon}{dR}\right), \tag{1.59}$$

and thus they differ in profile as long as $\phi^2 d\Upsilon/dR$ is not constant. In the next section it will be shown that the horizontal transfer of westerly angular momentum is proportional to $\phi^2 d\Upsilon/dR$ at any level, and thus the mean temperature and mean pressure profiles will differ only when there is a transfer of westerly angular momentum.

2. The radial transfer of angular momentum and heat and the derivation of the vorticity field

In order to throw some light on the choice of the function $d\Upsilon/dR$ in (1.52), it is useful to calculate the transfer in the minus r direction of westerly (i.e. in the direction of θ increasing) angular momentum (M_H) across a circle of radius r at a height z. This is clearly the integral from $\theta = 0$ to $\theta = 2\pi$ of the product $-\rho_0 r^* u_0 u_r$; hence

$$M_H = 4\rho_0 b^4 \omega^4 S^2 \int_{\theta=0}^{2\pi} \frac{\partial p}{\partial R}\frac{\partial p}{\partial \theta}\, d\theta. \tag{2.1}$$

The leading terms in the angular momentum transfer will arise from the geostrophic flow discussed in the previous section, and, bearing in mind the transformation leading to (1.20), it follows that in terms of P_0 in (1.58) we have

$$M_H = \rho_0 \kappa^2 m^{-2} \omega^2 \int_0^{2\pi} \frac{\partial P_0}{\partial R}\frac{\partial P_0}{\partial \theta}\, d\theta. \tag{2.2}$$

Using (1.58), we have

$$\frac{\partial P_0}{\partial \theta} = m\phi(R)\cos(m\theta - \Psi) - \frac{mB}{A}\phi(R)\sin(m\theta - \Psi), \tag{2.3}$$

$$\frac{\partial P_0}{\partial R} = \frac{\partial f}{\partial R} + \left(\frac{d\phi}{dR} + \frac{B}{A}\phi\frac{d\Psi}{dR}\right)\sin(m\theta - \Psi) + \left(\frac{B}{A}\frac{d\phi}{dR} - \phi\frac{d\Psi}{dR}\right)\cos(m\theta - \Psi), \tag{2.4}$$

and thus
$$M_H = -\rho_0 \kappa^2 m^{-1}\omega^2 \pi M, \tag{2.5}$$

where
$$M = \left(1 + \frac{B^2}{A^2}\right)\phi^2 \frac{d\Psi}{dR}. \tag{2.6}$$

We will now derive a general result concerning angular momentum transfer. The complete equation of motion in the direction of θ increasing relative to axes which are fixed in space is

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + w\frac{\partial v}{\partial z} + \frac{uv}{r} = \frac{1}{\rho}\left\{\frac{1}{r}\frac{\partial p_{\theta\theta}}{\partial \theta} + \frac{\partial p_{\theta z}}{\partial z} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 p_{r\theta})\right\}, \tag{2.7}$$

where $p_{\theta\theta}, p_{\theta z}, p_{r\theta}$ are the viscous stresses. The appropriate continuity equation is

$$\frac{\partial p}{\partial t} + \frac{\partial}{\partial r}(\rho r u) + \frac{\partial}{\partial \theta}(\rho v) + \frac{\partial}{\partial z}(\rho r w) = 0, \tag{2.8}$$

and using these two equations we can deduce the Reynolds stress form of (2.7), namely

$$\frac{\partial}{\partial t}(\rho v r^2) + \frac{\partial}{\partial r}(\rho u v r^2) + \frac{\partial}{\partial \theta}(\rho v^2 r) + \frac{\partial}{\partial z}(\rho v w r^2) = r\frac{\partial p_{\theta\theta}}{\partial \theta} + r^2\frac{\partial p_{\theta z}}{\partial z} + \frac{\partial}{\partial r}(r^2 p_{r\theta}). \tag{2.9}$$

We shall write m_H and m_V for the horizontal ($-r$-direction) and vertical ($+z$-direction) transports of westerly angular momentum (about $r = 0$) across a circle of radius r at height z, so that

$$m_H = -\int_0^{2\pi} \rho u v r^2\, d\theta, \tag{2.10}$$

$$m_V = \int_0^{2\pi} \rho w v r^2\, d\theta. \tag{2.11}$$

We multiply (2.9) by $dr\, d\theta\, dz$ and integrate over an annular volume τ bounded by $z_0, z_1\,(> z_0)$, r_0 and $r_1\,(> r_0)$, and we then obtain after some reduction

$$\frac{\partial}{\partial t}\left\{\int_\tau \rho v r\, d\tau\right\} + \int_{z_0}^{z_1}(m_H|_{r_0} - m_H|_{r_1})\,dz + \int_{r_0}^{r_1}(m_V|_{z_1} - m_V|_{z_0})\,dr$$
$$= \int_{r_0}^{r_1} dr \int_0^{2\pi} r^2(p_{\theta z}|_{z_1} - p_{\theta z}|_{z_0})\,d\theta + \int_{z_0}^{z_1} dz \int_0^{2\pi} (r_1^2 p_{r\theta}|_{r_1} - r_0^2 p_{r\theta}|_{r_0})\,d\theta. \tag{2.12}$$

In the present problem the time average value of the first term is zero and henceforth this term will be ignored. In this case, equation (2.12) states that the total flow of westerly angular momentum outwards across the boundary S of a volume τ is equal to the moment about $r = 0$ of the viscous stresses acting on the complete surface S. Viscosity has been ignored in the formulation of the problem of the first section; but since the influence of viscosity in this angular momentum result is equivalent to an appropriate supply of angular momentum across S, it is clear that we can simulate the effect of viscosity by an appropriate distribution of sources and sinks of angular momentum over S. I shall assume therefore that there are distributions of sources and sinks: (a) on the surface $z = \delta$ which is the plane upper boundary of the 'boundary layer' in $0 < z \leqslant \delta$, and (b) on the surfaces $r = b-\delta$, $r = a+\delta$ which are the cylindrical boundaries on the inside of the 'boundary layers' in $b-\delta \leqslant r < b$, $a < r \leqslant a+\delta$. The details of this distribution of sources and sinks in any given case can be obtained only by solving the complete Navier-Stokes equations, but here we shall assume that the distribution can be prescribed. Away from the boundary layers it is a good approximation that the total flow of angular momentum across the boundary S is zero, and assuming this to be exactly true we obtain the following result from (2.12):

$$\int_{z_0}^{z_1} (m_H\,|_{r_0} - m_H|_{r_1})\,dz + \int_{r_0}^{r_1} (m_V\,|_{z_1} - m_V|_{z_0})\,dr = 0; \tag{2.13}$$

and if we take $z_1 = z_0+\delta z$, $r_1 = r_0+\delta r$, we deduce that

$$\frac{\partial m_H}{\partial r} = \frac{\partial m_V}{\partial z}. \tag{2.14}$$

This will be assumed to be valid in the range $\delta < z < h$, $a+\delta < r < b-\delta$. If we introduce now the quantities M_H, M_V to represent the horizontal and vertical transports of westerly angular momentum for the relative flow (i.e. relative to axes which are rotating with angular velocity ω as in § 1), then it easily follows that

$$\frac{\partial M_H}{\partial r} = \frac{\partial M_V}{\partial z}. \tag{2.15}$$

Hence with the M_H defined in (2.5) the corresponding M_V will be given by

$$M_V = \rho_0 \int_0^{2\pi} r^{*2} u_\theta u_z\, d\theta \tag{2.16}$$

$$= -\rho_0 \kappa^2 m^{-1} \omega^2 \pi \frac{d}{dr}\left(\phi^2 \frac{d\Upsilon}{dR}\right) \int^z \left(1+\frac{B^2}{A^2}\right) dz. \tag{2.17}$$

It is clear now that if either M_H or M_V is prescribed on a plane $z =$ constant, then this is equivalent to postulating the behaviour of the unknown function Υ.

If $\phi^2(d\Upsilon/dR) =$ constant, it follows that $M_V = 0$, and thus the source of angular momentum can be taken at the cylindrical wall $r = b$ with an equal sink at $r = a$. If, on the other hand, there is no source or sink of angular momentum at the side walls, the whole of the angular momentum which emanates from one part of the base must be assumed to return to a sink in the remaining part of the base, so that

$$\int_a^b M_V\, dr = 0. \tag{2.18}$$

This type of angular momentum flow is the one which is similar to that of the atmosphere, and this case can be satisfied here provided

$$\left(\phi^2 \frac{d\Upsilon}{dR}\right)\bigg|_b = \left(\phi^2 \frac{d\Upsilon}{dR}\right)\bigg|_a. \tag{2.19}$$

We shall discuss these results further in § 4, and we now turn our attention to the radially inward transport of heat.

The radially inward transport of heat (H.T.) across a circle of radius r^* at height z^* will be obtained by integrating the product $\kappa u_r(\partial\tau^*/\partial r^*)$, hence

$$\text{H.T.} = -4\kappa\rho_0 b\omega^2 S(\Delta T)_H \int_0^{2\pi} \frac{\partial p}{\partial\theta}\frac{\partial\tau}{\partial r}\,d\theta. \tag{2.20}$$

The leading terms in this heat transfer will be given by

$$\text{H.T.} = -4\kappa\rho_0 b\omega^2 S(\Delta T)_H\,\epsilon^2 m^{-2}\int_0^{2\pi}\frac{\partial P_0}{\partial\theta}\frac{\partial T_0}{\partial r}\,d\theta,$$

and since, from (1.56),

$$\frac{\partial T_0}{\partial r} = \frac{\partial F}{\partial r}+\frac{1}{A}\frac{d\phi}{\partial r}\cos(m\theta-\Upsilon)+\frac{1}{A}\,\phi\frac{d\Upsilon}{dR}\sin(m\theta-\Upsilon), \tag{2.21}$$

it follows that

$$\text{H.T.} = -4\kappa\rho_0 b\omega^2 S\epsilon^2 m^{-1}(\Delta T)_H\,\pi H, \tag{2.22}$$

where

$$H = \frac{1}{A}\left\{\phi\frac{d\phi}{d\tau}-\frac{B}{A}\phi^2\frac{d\Upsilon}{dr}\right\}. \tag{2.23}$$

It will be seen that there is an essential difference between the angular momentum and heat transport expressions, both in their z and r variations, in particular the heat transport can exist even when the momentum transport is zero.

A third quantity which will be required later in the paper is the vorticity of the two-dimensional flow of the previous section; if this is denoted by ζ_0 then we have, working with the non-dimensional symbols,

$$\zeta_0 = \frac{\partial v_0}{\partial x}-\frac{\partial u_0}{\partial y} = \frac{\partial^2 p_0}{\partial x^2}+\frac{\partial^2 p_0}{\partial y^2}, \tag{2.24}$$

hence

$$r^2\zeta_0 = r^2\frac{\partial^2 p_0}{\partial r^2}+r\frac{\partial p_0}{\partial r}+\frac{\partial^2 p_0}{\partial\theta^2}$$

$$= \epsilon^2 m^{-2}\left(\frac{\partial^2 P_0}{\partial R^2}+\frac{\partial^2 P_0}{\partial\theta^2}\right). \tag{2.25}$$

Using the expression (1.58) for P_0, we obtain after some reduction

$$r^2\zeta_0 = \epsilon^2 m^{-2}\left\{\frac{\partial^2 f}{\partial R^2}+X\sin(m\theta-\Upsilon)+Y\cos(m\theta-\Upsilon)\right\}, \tag{2.26}$$

where

$$X = \frac{d^2\phi}{dR^2}-\phi\left(\frac{d\Upsilon}{dR}\right)^2-m^2\phi+\frac{B}{A}\left(\phi\frac{d^2\Upsilon}{dR^2}+2\frac{d\phi}{dR}\frac{d\Upsilon}{dR}\right), \tag{2.27}$$

$$Y = \frac{B}{A}\left\{\frac{d^2\phi}{dR^2}-\phi\left(\frac{d\Upsilon}{dR}\right)^2-m^2\phi\right\}-\left(\phi\frac{d^2\Upsilon}{dR^2}+2\frac{d\phi}{dR}\frac{d\Upsilon}{dR}\right). \tag{2.28}$$

We shall satisfy the Helmholtz vorticity equation with the vorticity function (2.24) when we proceed to the higher approximations in the next section (see (3.13)).

3. Higher order approximations and the determination of the vertical velocity

It is possible to determine the higher order approximations in this problem by taking power series in S for the various dependent variables taking as the first approximation the equations (1.14). Thus we shall take

$$u = u_0 + Su_1 + S^2u_2 + \dots, \tag{3.1}$$

$$v = v_0 + Sv_1 + S^2v_2 + \dots, \tag{3.2}$$

$$w = Sw_1 + S^2w_2 + \dots, \tag{3.3}$$

$$p = p_0 + Sp_1 + S^2p_2 + \dots, \tag{3.4}$$

$$\tau = \tau_0 + S\tau_1 + S^2\tau_2 + \dots, \tag{3.5}$$

where u_0, v_0, p_0, τ_0 are given by (1.14). If we substitute the above expressions in (1.8), (1.9), (1.10), (1.11) and (1.12) and equate the leading terms, (1.14) and (1.15) are obtained. Equating to zero the coefficients of the next power in S, we obtain the following equations:

$$\frac{du_0}{dt} - v_1 = -\frac{\partial p_1}{\partial x}, \tag{3.6}$$

$$\frac{dv_0}{dt} + u_1 = -\frac{\partial p_1}{\partial y}, \tag{3.7}$$

$$0 = -\frac{\partial p_1}{\partial z} + \tau_1, \tag{3.8}$$

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = 0, \tag{3.9}$$

$$u_0\frac{\partial \tau_1}{\partial x} + u_1\frac{\partial \tau_0}{\partial x} + v_0\frac{\partial \tau_1}{\partial y} + v_1\frac{\partial \tau_0}{\partial y} + w_1\frac{\partial \tau_0}{\partial z} = \epsilon\left(\frac{\partial^2 \tau_1}{\partial x^2} + \frac{\partial^2 \tau_1}{\partial y^2}\right), \tag{3.10}$$

where the term $\partial^2\tau_1/\partial z^2$ has been omitted in (3.10). From this set of five equations there are sufficient equations to solve for u_1, v_1, w_1, p_1 and τ_1. In order to determine w_1 however, it is necessary to use the three equations (3.6), (3.7) and (3.9).

The first equation (3.6) gives

$$\begin{aligned} v_1 &= \frac{\partial p_1}{\partial x} + u_0\frac{\partial u_0}{\partial x} + v_0\frac{\partial u_0}{\partial y} \\ &= \frac{\partial}{\partial x}\{p_1 + \tfrac{1}{2}u_0^2 + \tfrac{1}{2}v_0^2\} - v_0\zeta_0, \end{aligned} \tag{3.11}$$

where ζ_0 is defined in (2.24). Similarly (3.7) gives

$$\begin{aligned} u_1 &= -\frac{\partial p_1}{\partial y} - u_0\frac{\partial v_0}{\partial x} - v_0\frac{\partial v_0}{\partial y} \\ &= -\frac{\partial}{\partial y}(p_1 + \tfrac{1}{2}u_0^2 + \tfrac{1}{2}v_0^2) - u_0\zeta_0, \end{aligned} \tag{3.12}$$

and thus by substituting for u_1 and v_1 in (3.9) we obtain

$$\frac{\partial w_1}{\partial z} = \frac{\partial}{\partial x}(u_0\zeta_0) + \frac{\partial}{\partial y}(v_0\zeta_0),$$

so that, using (1.14), we have

$$\frac{\partial w_1}{\partial z} = u_0\frac{\partial \zeta_0}{\partial x} + v_0\frac{\partial \zeta_0}{\partial y};$$

hence the form of the Helmholtz vorticity equation is

$$\frac{\partial w_1}{\partial z} = \frac{\partial p_0}{\partial x}\frac{\partial \zeta_0}{\partial y} - \frac{\partial p_0}{\partial y}\frac{\partial \zeta_0}{\partial x}. \tag{3.13}$$

In polar co-ordinates this becomes

$$\frac{\partial w_1}{\partial z} = \frac{1}{r}\left\{\frac{\partial p_0}{\partial r}\frac{\partial \zeta_0}{\partial \theta} - \frac{\partial p_0}{\partial \theta}\frac{\partial \zeta_0}{\partial r}\right\},$$

and it may be verified that this equation can be written in the form

$$r^4\frac{\partial w_1}{\partial z} = \epsilon m^{-1}\left\{\frac{\partial P_0}{\partial R}\frac{\partial}{\partial \theta}(r^2\zeta_0) - \frac{\partial P_0}{\partial \theta}\frac{\partial}{\partial R}(r^2\zeta_0) + 2\frac{\partial P_0}{\partial \theta}(r^2\zeta_0)\right\},$$

so that by using (2.13) we have

$$\begin{aligned}\frac{r^4(\partial w_1/\partial z)}{\epsilon^3 m^{-4}} = &\left\{\frac{\partial f}{\partial R} + \left(\frac{d\phi}{dR} + \frac{B}{A}\phi\frac{d\Psi}{dR}\right)s_1 + \left(-\phi\frac{d\Psi}{dR} + \frac{B}{A}\frac{d\phi}{dR}\right)c_1\right\}\{Xc_1 - Ys_1\} \\ &-\left\{\frac{\partial^3 f}{\partial R^3} + \left(\frac{\partial X}{\partial R} + Y\frac{d\Psi}{dR}\right)s_1 + \left(\frac{\partial Y}{\partial R} - X\frac{d\Psi}{dR}\right)c_1\right\}\left\{\phi c_1 - \frac{B}{A}\phi s_1\right\} \\ &+2\left\{\frac{\partial^2 f}{\partial R^2} + Xs_1 + Yc_1\right\}\left\{\phi c_1 - \frac{B}{A}\phi s_1\right\},\end{aligned} \tag{3.14}$$

where we have written

$$s_r = \sin r(m\theta - \Psi), \quad c_r = \cos r(m\theta - \Psi). \tag{3.15}$$

In this expression for $\partial w_1/\partial z$ there will be terms independent of θ and it is clear that such terms will supply the first information concerning meridional cells in the general circulation pattern. These terms will arise from the c_1^2 and s_1^2 products and are as follows:

$$\begin{aligned}&-\tfrac{1}{2}Y\left(\frac{d\phi}{dR} + \frac{B}{A}\phi\frac{d\Psi}{dR}\right) + \tfrac{1}{2}X\left(-\phi\frac{d\Psi}{dR} + \frac{B}{A}\frac{d\phi}{dR}\right) - \tfrac{1}{2}\phi\left(\frac{\partial Y}{\partial R} - X\frac{d\Psi}{dR}\right) \\ &+\frac{1}{2}\frac{B}{A}\phi\left(\frac{\partial X}{\partial R} + Y\frac{d\Psi}{dR}\right) - \frac{B}{A}\phi X + \phi Y \\ &= \frac{1}{2}\frac{\partial}{\partial R}\left\{\phi\left(-Y + \frac{B}{A}X\right)\right\} - \phi\left(-Y + \frac{B}{A}X\right).\end{aligned}$$

Using (2.14) and (2.15) we have

$$\frac{B}{A}X - Y = \left(1 + \frac{B^2}{A^2}\right)\left(\phi\frac{d^2\Psi}{dR^2} + 2\frac{d\phi}{dR}\frac{d\Psi}{dR}\right), \tag{3.16}$$

and thus if we denote by $\overline{w}_1$ that part of w_1 which is independent of θ when we have

$$\frac{r^4(\partial \overline{w}_1/\partial z)}{\epsilon^3 m^{-4}} = \frac{1}{2}\frac{\partial}{\partial R}\left\{\left(1 + \frac{B^2}{A^2}\right)\frac{d}{dR}\left(\phi^2\frac{d\Psi}{dR}\right)\right\} - \left(1 + \frac{B^2}{A^2}\right)\frac{d}{dR}\left(\phi^2\frac{d\Psi}{dR}\right). \tag{3.17}$$

It may be noted that the right-hand side of this expression can be written entirely in terms of M which is defined in (2.6), and in this case we have the interesting result

$$\frac{r^4(\partial \bar{w}_1/\partial z)}{\epsilon^2 m^{-4}} = \frac{1}{2}\frac{\partial^2 M}{\partial R^2} - \frac{\partial M}{\partial R}, \tag{3.18}$$

which relates the meridional part of the vertical velocity with the horizontal angular momentum transfer. Using (1.40), an alternative form of (3.18) is

$$\frac{2r^4(\partial \bar{w}_1/\partial z)}{\epsilon^3 m^{-4}} = r^2\frac{\partial^2 M}{\partial r^2} - r\frac{\partial M}{\partial r}. \tag{3.19}$$

We may note that the meridional part of the vertical velocity will be identically zero if M shows no variation with r or if M is proportional to r^2. The remaining terms in the expression for $\partial w_1/\partial z$ do not simplify in the same way as the above meridional part and in general we have from (3.14)

$$\begin{aligned}\frac{r^4(\partial w_1/\partial z)}{\epsilon^3 m^{-4}} = \frac{1}{2}\frac{\partial^2 M}{\partial R^2} - \frac{\partial M}{\partial R} &+ \left\{X\frac{\partial f}{\partial R} - \phi\frac{\partial^3 f}{\partial R^3} + 2\phi\frac{\partial^2 f}{\partial R^2}\right\}\cos(m\theta - \Psi) \\ &- \left\{Y\frac{\partial f}{\partial R} - \frac{B}{A}\phi\frac{\partial^3 f}{\partial R^3} + \frac{2B}{A}\phi\frac{\partial^2 f}{\partial R^2}\right\}\sin(m\theta-\Psi) \\ &+ \frac{1}{2}\left\{\left(Y + \frac{B}{A}X\right)\frac{d\phi}{dR} - \phi\frac{\partial}{\partial R}\left(Y + \frac{B}{A}X\right) + 2\phi\left(Y + \frac{B}{A}X\right)\right\}\cos 2(m\theta - \Psi) \\ &+ \frac{1}{2}\left\{\left(X - \frac{B}{A}Y\right)\frac{d\phi}{dR} - \phi\frac{\partial}{\partial R}\left(X - \frac{B}{A}Y\right) + 2\phi\left(X - \frac{B}{A}Y\right)\right\}\sin 2(m\theta - \Psi).\end{aligned} \tag{3.20}$$

The terms which involve $\cos 2(m\theta - \Psi)$ and $\sin 2(m\theta - \Psi)$ are in general non-vanishing, thus w_1 will consist of a meridional part, a part which varies periodically with the wave and a part which varies periodically with half the fundamental wave length. To derive w_1 we can assume that $w_1 = 0$ at $z = 0$ then w_1 is the integral from 0 to z of the right-hand side of (3.20).

4. Particular examples of the foregoing theory

We consider first the boundary conditions. Since molecular viscosity is ignored, the condition of zero normal velocity has to be satisfied at each of the bounding solid surfaces. Therefore $\partial p/\partial\theta$ must vanish at the cylindrical walls, in other words the function ϕ in (1.58) must be zero at these boundaries. In the non-dimensional scheme in (1.6), the quantity b can be treated as the radius of the outer cylinder, and thus $r^* = b$ of $r = 1$ will represent the outer cylinder. If a is the radius of the inner cylinder, then $r = a/b$ will represent the inner cylinder. In terms of R introduced in (1.40) the outer and inner boundaries will be $R = 0$ and $R = -R_0$ respectively, where $R_0 = \log_e(b/a)$. Thus, the velocity conditions upon $\phi(R)$ reduce to the following:

$$\phi(0) = \phi(-R_0) = 0. \tag{4.1}$$

It is clear from (1.56) that the conditions of constancy of temperature will be satisfied on the two cylindrical boundaries due to (4.1). The only remaining condition upon the temperature field is that the difference in temperature from one cylinder to the other must be $(\Delta T)_H$ at some level. It follows therefore that

$$\tau\,|_{R=0} - \tau\,|_{R=-R_0} = 1. \tag{4.2}$$

In terms of F in (1.56) we must therefore have, at a particular z value,

$$F\,|_{R=0} - F\,|_{R=-R_0} = \frac{m}{\epsilon}. \tag{4.3}$$

Up to the present the function $A(z)$, introduced originally in (1.33), has been left arbitrary. It will be seen from (1.51) that $\partial F/\partial R$ is proportional to $1/A$ and that the heat flow at the cylindrical boundaries, namely $\partial F/\partial r$ is also proportional to $1/A$. In order to make the problem as simple as possible, it is proposed to take uniform heat flow over the cylindrical boundaries since there is no guidance from the experiment in this matter, thus with no loss of generality we shall take

$$A = 1, \tag{4.4}$$

in which case the constant j in (1.51) is a measure of the heat flow across a section of unit length of either cylinder. Accordingly, from (1.54), we have

$$B = z + \beta, \tag{4.5}$$

where β is an arbitrary constant. With this choice of A, the solutions (1.56) and (1.58) for T_0 and P_0 become

$$T_0 = F + \phi(R)\cos(m\theta - \Psi), \tag{4.6}$$

$$P_0 = f + \phi(R)\sin(m\theta - \Psi) + (\beta + z)\,\phi(R)\cos(m\theta - \Psi). \tag{4.7}$$

The third term of P_0 evidently becomes increasingly important as z increases, hence the phase difference of the temperature and pressure fields diminishes with increasing height. The constant β is a measure of the difference in phase of the temperature and pressure waves, and the value of β, here arbitrary, can be derived from the experiment; but whether β be positive or negative, the pressure wave will be ahead of the temperature wave at all levels, the departure of the two waves diminishing with increasing height. Thus, the pressure wave has a 'backward tilt', i.e. to the west as in the case of atmospheric troughs. We may also observe that the angular momentum transfer (2.6) will have a parabolic distribution in the vertical when (4.4) and (4.5) are true, and the maximum transfer will take place at the free surface.

It is worth while noting also that we may draw a parallel with the atmosphere where the heat source in the lowest layers (troposphere) is at the equator and in the upper layers (stratosphere) at high latitudes by taking

$$\frac{1}{A} = (z_0 - z)\,e^{-\lambda z}.$$

With this choice of A, equation (1.54) gives for B the value

$$B = -\frac{1}{\lambda} + \left(\beta e^{\lambda z} + \frac{1}{\lambda^2}\right)\Big/(z_0 - z).$$

We note that B has an infinity at the position where $1/A$ changes its sign, that is at the tropopause, and the pressure wave becomes 180° out of phase with the temperature wave in moving through this level. This will be avoided only if β is chosen appropriately.

Returning now to (4.3) and using (4.4), it then follows that the appropriate form of this condition is

$$\int_{R=-R_0}^{R=0} \frac{\partial F}{\partial R} dR = \int_{-R_0}^{0} \{j - \tfrac{1}{2}\phi^2(R)\} dR = \frac{m}{\epsilon}, \tag{4.8}$$

that is

$$jR_0 = \frac{m}{\epsilon} + \frac{1}{2}\int_{-R_0}^{0} \phi^2(R)\, dR. \tag{4.9}$$

Case I. Zero momentum transport; $d\Psi/dR = 0$.

This case may not have much practical significance, but so many of the technical difficulties encountered in general are also met in this simple case that it is worth while devoting considerable attention to it. The equation (1.52) now becomes

$$\frac{d^2\phi}{dR^2} = (m^2 - j)\,\phi + \tfrac{1}{2}\phi^3. \tag{4.10}$$

The first integral of (4.10) is

$$\left(\frac{d\phi}{dR}\right)^2 = (m^2 - j)\,\phi^2 + \tfrac{1}{4}\phi^4 + C, \tag{4.11}$$

where C is an arbitrary constant of integration. The function ϕ vanishes at the two end-points of the range $-R_0 \leqslant R \leqslant 0$; and, assuming that ϕ is a continuous function, it is necessary that $d\phi/dR$ should vanish within the R range. If we suppose that $d\phi/dR$ vanishes when $\phi = \phi_1$, then

$$0 = (m^2 - j)\,\phi_1^2 + \tfrac{1}{4}\phi_1^4 + C, \tag{4.12}$$

and thus

$$\left(\frac{d\phi}{dR}\right)^2 = \tfrac{1}{4}(\phi_1^2 - \phi^2)\,(4j - 4m^2 - \phi_1^2 - \phi^2). \tag{4.13}$$

Clearly it is necessary that

$$4j - 4m^2 - \phi_1^2 > 0 \tag{4.14}$$

in order for $(d\phi/dR)^2$ to be positive near $\phi = 0$; hence j is positive, or alternatively ϕ_1^2 possesses an upper bound $4(j - m^2)$. If we write

$$\phi_2^2 = 4j - 4m^2 - \phi_1^2, \tag{4.15}$$

then

$$\left(\frac{d\phi}{dR}\right)^2 = \tfrac{1}{4}(\phi_1^2 - \phi^2)\,(\phi_2^2 - \phi^2). \tag{4.16}$$

We shall take

$$\phi_2^2 > \phi_1^2, \quad k^2 = \frac{\phi_1^2}{\phi_2^2}, \tag{4.17}$$

since there is no loss of generality in doing this, and ϕ_1 will be taken to be positive. If we write

$$\phi = \phi_1 y, \quad x = \tfrac{1}{2}R\phi_2, \tag{4.18}$$

then (4.16) becomes

$$\left(\frac{dy}{dx}\right)^2 = (1 - y^2)\,(1 - k^2y^2), \tag{4.19}$$

where k^2 is defined in (4.17) and $0 < k^2 < 1$. The solution of (4.19) which vanishes when $R = 0$ or $x = 0$ is the Jacobian elliptic function $y = \text{sn}\,(x, k)$, and hence the solution of (4.16) which vanishes at $R = 0$ is

$$\phi = \phi_1 \,\text{sn}\{-\tfrac{1}{2}R\phi_2, k\}, \tag{4.20}$$

where a slight adjustment in sign has been made in order to make $\phi > 0$ when $-R_0 < R < 0$. This elliptic function has an infinite number of zeros since we have

$$\text{sn}\,(u+2K, k) = -\text{sn}\,(u, k),$$
$$\text{sn}\,(u+4Kn, k) = \text{sn}\,(u, k) \quad (n = 0, 1, 2\ldots).$$

If that function $\phi(R)$ is chosen which vanishes at the points $R = 0$ and $R = -R_0$ and has one simple maximum in the range $-R_0 \leqslant R \leqslant 0$, then we must have

$$\tfrac{1}{2}R_0\phi_2 = 2K; \tag{4.21}$$

but there are an infinite set of solutions for ϕ, the one which possesses one maximum and one minimum being the case $\frac{1}{2}R_0\phi_2 = 4K$, and the general case is $\frac{1}{2}R_0\phi_2 = 2nK$. These different modes of solution are of course not superposable since the ϕ equation is non-linear and only one of them can exist at any one time. We shall concentrate here on the main mode which is given by (4.21). For this main mode of ϕ with a single maximum $\phi = \phi_1$ at $\frac{1}{2}R_1\phi_2 = K$ where $\text{sn}\,(K, k) = 1$, it follows that the position of the maximum value is such that $R_1 = \frac{1}{2}R_0$; in terms of r, this indicates that the maximum is at r_1, where $r_1 = r_0^{\frac{1}{2}} = (a/b)^{\frac{1}{2}}$, and in terms of the original variables at $r_1^* = (ab)^{\frac{1}{2}}$, the geometric mean of the radii of the two cylinders.

The quantity K which appears in (4.21) is a function of k, and, when k is known, K can be found from tables. Since $k = \phi_1/\phi_2$ and ϕ_2 is a function of j (4.15), it follows that (4.21) is a relation between j and ϕ_1. We may observe that, since $\phi_1 < \phi_2$, then from (4.15) we obtain

$$\phi_1^2 < 2j - 2m^2 < \phi_2^2. \tag{4.22}$$

If we use (4.9), we obtain a second relation between j and ϕ_1, namely

$$jR_0 = \frac{m}{\epsilon} + \tfrac{1}{2}\phi_1^2 \int_{-R_0}^{0} \text{sn}^2\,(-\tfrac{1}{2}R\phi_2, k)\,dR$$
$$= \frac{m}{\epsilon} + \frac{\phi_1^2}{\phi_2}\int_0^{2K} \text{sn}^2\,(x, k)\,dx;$$

hence

$$jR_0 = \frac{m}{\epsilon} + 2k\phi_1 \int_0^K \text{sn}^2\,(x, k)\,dx. \tag{4.23}$$

The complete elliptic integral E of the second kind is defined by

$$E = \int_0^K \text{dn}^2\,(x, k)\,dx, \tag{4.24}$$

and since $\text{dn}^2\,(x, k) = 1 - k^2\,\text{sn}^2(x, k)$, it follows

$$\int_0^K \text{sn}^2\,(x, k)\,dx = \frac{K-E}{k^2}; \tag{4.25}$$

hence, from (4.23),

$$jR_0 = \frac{m}{\epsilon} + \frac{2\phi_1}{k}(K-E). \tag{4.26}$$

Equations (4.15), (4.17), (4.21) and (4.26) now represent four independent equations between ϕ_1, ϕ_2, k and j and ϵ. Hence, this can be regarded as a one-parameter problem in the sense that if one of these five quantities is defined then, in theory, the other four can be determined; the quantity ϵ is regarded as a parameter since the quantity ω is the angular velocity of the wave system (not the angular velocity of the apparatus) and is not therefore a prescribed quantity. It is probably most convenient to treat k as the parameter at our disposal in the range $0 < k < 1$. In terms of k we have

$$\phi_2 = \frac{4K}{R_0}, \quad \phi_1 = \frac{4kK}{R_0}, \tag{4.27}$$

$$j = m^2 + \frac{4K^2}{R_0^1}(1+k^2), \tag{4.28}$$

$$\frac{2\omega S b^2}{K} = \frac{1}{\epsilon} = mR_0 + \frac{4K\{2E-K(1-k^2)\}}{mR_0}. \tag{4.29}$$

Using these relations it is possible to throw some light on the conditions necessary for the existence of a particular wave number; their usefulness, however, is much reduced by the fact that the angular velocity ω in (4.29) is the angular velocity of the wave system with respect to fixed space axes and not the angular velocity ω_s of the apparatus as a whole. The difference between ω and ω_s is small and of the order S.

We consider first equation (4.29) which is of the form

$$\omega S = \frac{K}{2b^2}\left\{mR_0 + \frac{\mu^2}{mR_0}\right\}, \tag{4.30}$$

where $\mu^2 = 4K\{2E-K(1-k^2)\}$. For a given $R_0 (= \log b/a)$, a given m and a given k this relation between ω and S in the (ω, S)-plane is a rectangular hyperbola which we denote by Γ. As k varies from 0 to 1, it is easily shown that μ^2 moves from a minimum value of π^2 when $k = 0$ and increases monotonically to $+\infty$ as k tends to 1. Thus, if we let Γ_m denote the rectangular hyperbola

$$\omega S = \frac{K}{2b^2}\left\{mR_0 + \frac{\pi^2}{mR_0}\right\}, \tag{4.31}$$

it follows that as k moves from 0 to 1 the curve Γ in the (ω, S)-plane sweeps out that infinite area $\Gamma > \Gamma_m$ which lies on the side of the curve Γ_m remote from the origin. Since the minimum value of $mR_0 + (\pi^2/mR_0)$, considered as a function of mR_0, is attained when $mR_0 = \pi$ and its value is then 2π, it follows that the curves $\Gamma_1, \Gamma_2, \Gamma_3, \ldots$, corresponding to $m = 1, 2, 3, \ldots$, will move nearer to the origin until that integral value of $m = m^*$ is attained which makes $m^* R_0$ nearest π, and for m values greater than m^* the curves Γ_m will recede from the origin. For example, with $b = 2a$ ($R_0 = 0{\cdot}6931$), the successive curves Γ_s ($s = 1, 2, \ldots, 5$), approach the origin successively, but from $m = 6$ onwards the curves recede from the origin.

If we consider one of these curves, say Γ_1, then given a value of $\omega = \omega'$ and of $S = S'$ we can deduce that if the representative point (ω', S') is on the origin side of the curve Γ_1, then this particular wave number cannot exist with such a rotation and such a Rossby number. Alternatively, if the point (ω', S') is on the side of the curve Γ_1 remote from the origin, then there will be a unique k value in the range $0 < k < 1$ which will define the strength of the corresponding wave. A similar argument will apply to every curve $\Gamma_2, \Gamma_3, \ldots,$ of the family Γ_m, and we then have the following overall results:

(*a*) wave number $m = 1$ cannot exist if the point (ω, S) lies on the origin side of the curve Γ_1;

(*b*) wave numbers $m = 1$, $m = 2$ cannot exist if the point (ω, S) lies on the origin side of the curve Γ_2;

(*c*) wave numbers $m = 1$, $m = 2$, $m = 3$ cannot exist if the point (ω, S) lies on the origin side of the curve Γ_3;

and so on until that particular value of $m = m^*$ is attained which makes $m^* R_0$ nearest π; and for this case and beyond we have

(*d*) no wave numbers can exist if the point (ω, S) lies on the origin side of the curve Γ_{m^*}.

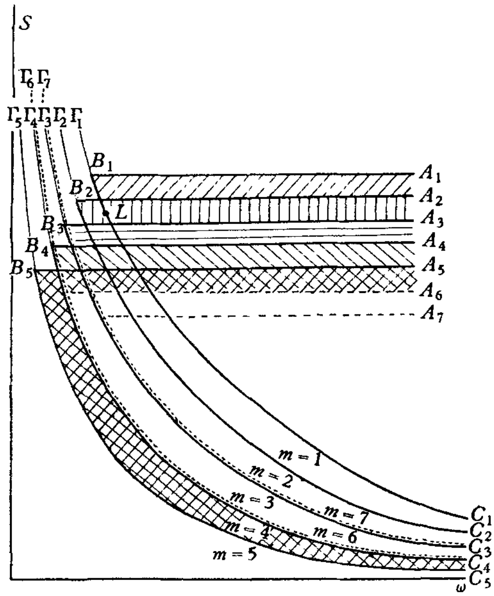

FIGURE 2. A schematic theoretical stability diagram.

In this final case (*d*) the theory does not give any indication of the particular flow pattern which will occur, but it may be inferred that the flow must be the spiral (or symmetric Hadley) type. The results stated in (*a*) to (*d*) above are summarized in diagrammatic form in figure 2.

It is of considerable interest to insert in figure 2 also the results concerning the existence of waves which were obtained by the present author in a stability investigation (1956). These results apply to infinitesimal waves, and the theory in that paper indicates that a wave number m can exist only if the Rossby number is less than a certain critical value. For convenience, table 5 of the earlier paper is reproduced below and expressed in terms of the parameters of this paper, the Rossby number Ro_T^* of the earlier paper being related† to the present S by the formula $S = \frac{1}{2}(1-a/b)\mathrm{Ro}_T^*$ and table 1 refers to the case $b = 2a$. This table is interpreted as follows: spiral flow will exist if $S > 0{\cdot}055$, wave number $m = 1$ can also exist if $S > 0{\cdot}05$, spiral flow and wave numbers $m = 1, 2$ can exist if $S > 0{\cdot}03$ and so on. When we insert the lines $S = 0{\cdot}055$, $S = 0{\cdot}05$; $S = 0{\cdot}03$, and so on, in figure 2 we can interpret the diagram in the following way. On the origin side of the curve $A_1B_1C_1$, the wave number $m = 1$ cannot exist; on the origin side of the curve $A_2B_2C_2$ the wave number $m = 2$ cannot exist, and so on. At such a point as P in this diagram it would then appear that any one of the wave patterns $m = 1$, $m = 2$ or $m = 3$ can exist; other factors must clearly enter into the discrimination between these possible wave patterns since only one wave is known to exist at any one time.

(Mean temperature 21 °C, $\alpha = 2{\cdot}1 \times 10^{-4}$)

Wave number m	0	1	2	3	4	5
Critical value of S	0·07	0·055	0·05	0·03	0·02	0·01

TABLE 1. Annulus case $b = 2a$

We now consider the heat transported by a particular wave pattern, this being equal to the heat flow across either of the cylindrical boundary walls. The heat flow per unit axial length across $r^* = b$ is given by

$$Q = \rho_0 c_p \kappa \int_0^{2\pi} \frac{\partial \tau^*}{\partial r^*} r^* \bigg|_{r^*=b} d\theta = \rho_0 c_p \kappa \frac{\epsilon}{m} (\Delta T)_H \int_0^{2\pi} \frac{\partial T}{\partial R}\bigg|_{R=0} d\theta.$$

Using (1.51) and also the boundary condition $\phi = 0$ at $R = 0$, we obtain

$$Q = Q(m,k) = \frac{2\pi\kappa\rho_0 c_p j\epsilon(\Delta T)_H}{mA(z)}$$

$$= 2\pi\kappa\rho_0 c_p m^{-1}(\Delta T)_H \epsilon \left\{m^2 + \frac{4\kappa^2(1+k^2)}{R_0^2}\right\}.$$

The Nusselt number Nu (Hide 1958) for this problem may be defined by

$$\mathrm{Nu} = \frac{(b-a)\,Q}{2\pi b\kappa\rho_0 c_p(\Delta T)_H}; \tag{4.32}$$

and thus in the case $A = 1$ we have

$$\mathrm{Nu} = \frac{(1-a/b)}{R_0}\,\frac{m^2R_0^2 + 4K^2(1+k^2)}{m^2R_0^2 + 4K\{2E - K(1-k^2)\}}. \tag{4.33}$$

It is easily shown that $(1-a/b)/R_0$ is the Nusselt number for an annulus of solid but uniform material when the cylindrical boundaries are subject to a differential temperature; and if this basic Nusselt number is denoted by Nu*, it follows that

† The small error due to the angular velocity definitions being slightly different will be ignored here.

the presence of liquid in the annulus has the effect of increasing the effective Nusselt number in the ratio

$$\frac{\mathrm{Nu}}{\mathrm{Nu}^*} = \frac{m^2R_0^2+4K^2(1+k^2)}{m^2R_0^2+4K\{2E-K(1-k^2)\}}. \tag{4.34}$$

If k is near zero we have $\mathrm{Nu} \doteqdot \mathrm{Nu}^*$, but if k is near its upper limit of unity we have $K \sim \log(4/\surd(1-k^2))$, $E \sim 1$, so that

$$\frac{\mathrm{Nu}}{\mathrm{Nu}^*} \sim \frac{K}{E} = \log\left(\frac{4}{\surd(1-k^2)}\right). \tag{4.35}$$

Thus the heat flow can be increased indefinitely if k can be made to tend to unity.

It is possible to use this concept of heat transport in order to help discriminate between different regions in the (ω, S)-plane. Consider the point L in the (ω, S)-plane which lies on Γ_1, and which is in the zone between A_2B_2 and A_3B_3. At this point, waves $m = 1$ and $m = 2$ are possible. Reckoned relative to the curve Γ_1, the value of k at L is 0, and the associated Nu for this point will be

$$(\mathrm{Nu})_{L,m=1} = \mathrm{Nu}^*.$$

At this same point the value of k reckoned relative to the curve Γ_2 is k_2, where

$$\mu^2(k_2) = 2(\pi^2-R_0^2),$$

and the corresponding value of Nu is

$$(\mathrm{Nu})_{L,m=2} = \mathrm{Nu}^*\frac{4R_0^2+4K^2(k_2)(1+k_2^2)}{4R_0^2+4K(k_2)[2E(k_2)-K(k_2)(1-k_2^2)]} > \mathrm{Nu}^*.$$

Hence, the $m = 2$ wave at L is capable of transporting more heat than the $m = 1$ wave. If one then assumes that the preferred liquid motion is always that one which transports the greatest amount of heat, then the point L will correspond to the $m = 2$ wave. A similar argument will apply to any point in the restricted zone between A_2B_2 and A_3B_3, and consequently this zone will correspond to the $m = 2$ wave. Kuo (1957), in a stability investigation, discusses this same zone between A_2B_2 and A_3B_3 and arrives at a similar conclusion using a different method, namely that in moving across A_2B_2 into this zone the $m = 2$ wave is the one which has maximum growth rate.

On this basis the whole of the (ω, S)-plane can be subdivided into definite regions where one can expect symmetric flow, wave number 1, wave number 2 and so on, and the complete picture is shown in figure 2.† We can look upon the lines A_1B_1, A_2B_2, ... as lines of dynamical instability, that is where the infinitesimal amplitude waves successively become resonated, and upon the curves $\Gamma_1, \Gamma_2, \ldots$ as curves of heating instability where the liquid has to change its wave pattern due to the amount of heat available being excessive or insufficient to maintain a particular wave form.

If we apply the heat transport argument to a point in the (ω, S)-plane which lies on Γ_6 and is in the zone between A_6B_6 and A_7B_7 (we deal specifically here with the

† The scales used in the schematic diagram (figure 2) may be inferred from the fact that the right-hand side of (4.30) has the order of magnitude 10^{-5}.

case $b = 2a$), it follows that, since Γ_6 is on the side of Γ_5 remote from the origin, the heat transport associated with $m = 6$ is less than that associated with $m = 5$. Thus, in such a geometrical configuration the maximum possible amount of heat can be transported by the steady wave $m = 5$, and beyond this m value the steady waves become less efficient as transporters of heat. It is of course not possible by this argument to prove that wave numbers 6, 7, 8, ..., cannot occur, but they are certainly less efficient as transporters of heat than is wave number 5.

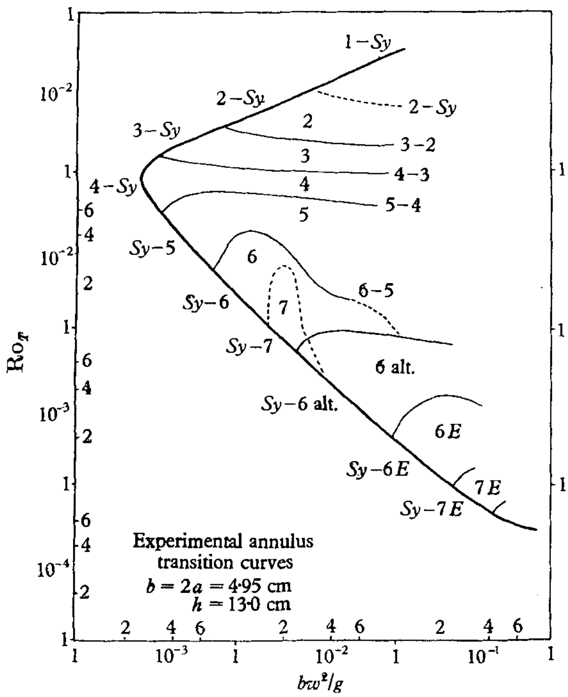

Figure 3. Experimental annulus transition curves obtained by Fultz.

We may compare figure 2 with one recently prepared by Fultz for the case $b = 2a$. The ordinate in the Fultz diagram, figure 3, is the Rossby number and the abscissa is a quantity proportional to ω_8^2; and it will be noted that there is a broad overall agreement between the two diagrams. There is a considerable complexity in the Fultz diagram for the lowest Rossby numbers where wave numbers 6 and 7 appear, and Fultz has described this region as containing types of instability which are inherently complicated. It is significant that this region of complicated wave features is associated in the theoretical investigation with the heat transport attaining its maximum value on Γ_5. Assuming that the value m^* gives an indication of the maximum number of steady waves in any given geometrical configuration, we have in general

$$m^* = \left[\frac{\pi}{R_0}\right] = \left[\frac{\pi}{\log(b/a)}\right], \tag{4.36}$$

where $[x]$ denotes that integral value which is closest to x. As b approaches a, $\log(b/a)$ tends to zero and m^* becomes large (this is observed experimentally; up to 15 waves have been produced in a sufficiently narrow annulus); as a tends to zero and lies in the range $0 < a < b\,e^{-\pi}$, it would appear that no wave motion is then possible.

For moderate values of k the typical magnitude of Nu from (4.33) is 2 or 3; Fultz in a private communication gives values of Nu which are of order 30 and over. It is clear therefore that the process of molecular conduction as assumed here cannot be the correct one for the transfer of heat in this problem. It is possible of course to postulate the existence of an eddy conductivity in order to achieve the correct order of magnitude for the heat transfer, but the more correct conclusion is probably that the bulk of the heat transfer is effected by the ageostrophic flow. In this same communication, Fultz states that 'there is a faint indication of an increase of Nu with m or some possibility of a maximum for intermediate m; it is also probable that there might be local maxima in the regions of strong vacillation'. This is in qualitative agreement with the picture described above, and it is possible to accept the present theory in as much as it provides insight into the mechanism of the change-over from one wave pattern to another. In this respect also we may compare (4.36) with an experimental result due to Hide, namely

$$\frac{m^*}{\pi}\left(\frac{b-a}{b+a}\right) = 0{\cdot}67. \tag{4.37}$$

Since
$$\log\left(\frac{b}{a}\right) = 2\left\{\left(\frac{b-a}{b+a}\right)+\frac{1}{3}\left(\frac{b-a}{b+a}\right)^3+\dots\right\},$$

it follows from (4.36) that we have approximately

$$\frac{m^*}{\pi}\left(\frac{b-a}{b+a}\right)\left\{1+\frac{1}{3}\left(\frac{b-a}{b+a}\right)^2+\dots\right\} = 0{\cdot}5;$$

and thus this also provides some justification for accepting (4.36) as the upper limit of the wave number.

The function ϕ being known, the functions F and f can now be determined. Using (4.4) and (1.51), we have

$$F = jR - \tfrac{1}{2}\phi_1^2\int_{-R}^{0} \mathrm{sn}^2\{-\tfrac{1}{2}R\phi_2, k\}\,dR, \tag{4.38}$$

there being no loss of generality in making $F = 0$ for $R = 0$. This gives the mean temperature field in the liquid. The mean pressure field f is given by (1.53); and in the present case $d\Psi/dR = 0$, we have

$$\frac{\partial f}{\partial R} = (\beta+z)(j-\tfrac{1}{2}\phi^2),$$

and
$$f = (\beta+z)\left\{jR-\tfrac{1}{2}\phi_1^2\int_{-R}^{0}\mathrm{sn}^2\{-\tfrac{1}{2}R\phi_2, k\}\,dR\right\}+\text{constant}. \tag{4.39}$$

As mentioned earlier the profiles of mean temperature and mean pressure at any height z are identical.

Since
$$j-\tfrac{1}{2}\phi_1^2 = m^2+\frac{4k'^2K^2}{R_0^2}$$

from (4.27) it follows that $\partial F/\partial R$ is always positive; and thus the mean temperature and mean pressure therefore increase monotonically with r. The minimum gradient will be encountered at $r^* = (ab)^{\frac{1}{2}}$ and the maxima of the gradient and therefore the maximum mean zonal velocity at the cylindrical walls. The action of viscosity in practice may invalidate this result near the bounding walls of the annulus.

Case II. Momentum transport proportional to $\phi^2(R)$; $d\Upsilon/dR$ = constant

In this case we shall assume that

$$\frac{d\Upsilon}{dR} = m\lambda, \tag{4.40}$$

where λ is a constant. This implies that $\Upsilon = m\lambda \log r$, and also, from (2.5) and (2.6), that the angular momentum transport is proportional to $\phi^2(R)$. Since ϕ vanishes at the bounding cylindrical walls and attains a maximum in the liquid between, it follows that the angular momentum shows a similar type of profile, and in order to produce such a horizontal transfer there will be in this case a mechanism supplying or withdrawing westerly angular momentum in the vertical direction. The equation (1.52) in this case becomes

$$\frac{dR^2}{d^2\phi} = (m^2+m^2\lambda^2-j)\,\phi+\tfrac{1}{2}\phi^3; \tag{4.41}$$

and, comparing (4.41) with (4.10), it is clear that the only difference between Case I and Case II lies in the coefficient of ϕ and that all the results pertaining to this case can be deduced from Case I provided m^2 is replaced by $m^2(1+\lambda^2)$. Thus, the formulae which replace (4.27) and which apply to this case will be

$$\left.\begin{aligned} \phi_2 = \frac{4K}{R_0}, \quad \phi_1 = \frac{4kK}{R_0}, \\ j = m^2(1+\lambda^2)+\frac{4K^2}{R_0^2}(1+k^2), \\ \frac{1}{\epsilon} = mR_0(1+\lambda^2)+\frac{4K}{mR_0}\{2E-K(1-k^2)\}. \end{aligned}\right\} \tag{4.42}$$

The formula for F, the mean zonal temperature, will be the same as (4.29), but the formula for $\partial f/\partial R$, given by (1.53), is in this case

$$\frac{\partial f}{\partial R} = (\beta+z)\,(j-\tfrac{1}{2}\phi^2)+2m\lambda\frac{d}{dR}(\log\phi). \tag{4.43}$$

The solution for f, the mean zonal pressure, will be

$$f = (\beta+z)\left\{jR-\tfrac{1}{2}\phi_1^2\int_{-R}^{0}\operatorname{sn}^2(-\tfrac{1}{2}R\phi_2,k)\,dR\right\}+2m\lambda\log\phi, \tag{4.44}$$

and, since ϕ vanishes at the two bounding walls it will be seen that the mean zonal pressure and mean zonal velocity have singularities at the boundaries.

Case III. Momentum transfer proportional to ϕ^3; $d\Upsilon/dR = \gamma\phi(R)$

One of the objections to the previous case is that the mean zonal pressure and velocity fields have singularities at the cylindrical bounding walls. Here we investigate a solution which does not suffer from this disadvantage. The formula (1.53) for $\partial f/\partial R$ indicates that the singularity in the zonal velocity arises from the term $\frac{1}{\phi^2}\frac{d}{dR}\left(\phi^2\frac{d\Upsilon}{dR}\right)$ on the right-hand side; and provided $d\Upsilon/dR$ has a simple zero at each of the bounding walls, the singularity will not arise. If, for example, Υ is such that

$$\frac{d\Upsilon}{dR} = \gamma\phi^s, \tag{4.45}$$

where $s \geqslant 1$, then the singularity of $\partial f/\partial R$ is avoided. We consider here the particular case $s = 1$, and shall take

$$\frac{d\Upsilon}{dR} = \gamma\phi \quad (\gamma = \text{constant}); \tag{4.46}$$

but the best possible value of s to choose must await a closer comparison with experiment. Whatever the deficiencies of this choice it has the great merit that the problem can be completely solved in this case and thus we can find what modifications are introduced into the solution by the presence of an angular momentum transfer.

Equation (1.52) for ϕ now becomes

$$\frac{d^2\phi}{dR^2} = (m^2-j)\,\phi + (\gamma^2+\tfrac{1}{2})\,\phi^3, \tag{4.47}$$

and this case can, as before, be solved in terms of elliptic functions. The first integral of (4.47) is

$$\left(\frac{d\phi}{dR}\right)^2 = (m^2-j)\,\phi^2 + (\tfrac{1}{4}+\tfrac{1}{2}\gamma^2)\,\phi^4 + \text{constant},$$

and if we suppose that $d\phi/dR$ vanishes when $\phi = \phi_1$, it follows that this equation can be expressed in the form

$$\left(\frac{d\phi}{dR}\right)^2 = \left(\frac{1+2\gamma^2}{4}\right)(\phi_1^2-\phi^2)(\phi_2^2-\phi^2), \tag{4.48}$$

where

$$\phi_2^2 = \frac{4(j-m^2)}{1+2\gamma^2} - \phi_1^2 > 0. \tag{4.49}$$

As before there is no loss of generality in taking $\phi_2 > \phi_1$, and we shall choose $k = \phi_1/\phi_2$, where $0 < k < 1$. The appropriate solution of (4.49) is

$$\phi = \phi_1\,\mathrm{sn}\{-\tfrac{1}{2}R\phi_2\sqrt{(1+2\gamma^2)}, k\}, \tag{4.50}$$

and this will satisfy both boundary conditions (4.1) provided

$$\tfrac{1}{2}R_0\phi_2\sqrt{(1+2\gamma^2)} = 2K. \tag{4.51}$$

The solution (4.50) has its maximum at the geometrical mean value $r^* = (ab)^{\frac{1}{2}}$; thus, this result is not influenced by momentum transport. The condition (4.9) becomes

$$jR_0 = \frac{m}{\epsilon} + \frac{2\phi_2}{\sqrt{(1+2\gamma^2)}}(K-E), \tag{4.52}$$

and thus the four equations which generalize the set (4.27) to (4.29) are as follows:

$$\phi_2 = \frac{4K}{R_0\sqrt{(1+2\gamma^2)}}, \quad \phi_1 = \frac{4Kk}{R_0\sqrt{(1+2\gamma^2)}}, \tag{4.53}$$

$$j = m^2 + \frac{4K^2(1+k^2)}{R_0^2}, \tag{4.54}$$

$$\frac{m}{\epsilon} = m^2 R_0 + \frac{\nu^2}{R_0}, \tag{4.55}$$

where

$$\nu^2 = \frac{4K}{1+2\gamma^2}\{2E - K[(1-k^2) - 2\gamma^2(1+k^2)]\}. \tag{4.56}$$

These equations can be interpreted as in the first case. It is easily shown that the quantity ν^2 defined in (4.56) increases monotonically from the value π^2 at $k = 0$ to infinity as k tends to unity, and thus it follows that all the detailed features of the stability diagram are not influenced by momentum transfer, in particular the result (4.36) for the maximum number of waves remains unchanged.

The formula for the mean temperature field is similar to the previous case, and is given by

$$F = jR - \tfrac{1}{2}\phi_1^2 \int_{-R}^{0} \text{sn}^2\{-\tfrac{1}{2}R\phi_2\sqrt{(1+2\gamma^2)}, k\}\, dR. \tag{4.57}$$

The mean zonal velocity field $\partial f/\partial r$ can be deduced from $\partial f/\partial R$ using (1.53) and (4.37), which give

$$\frac{\partial f}{\partial R} = \frac{B}{A}(j - \tfrac{1}{2}\phi^2) - 3\gamma \frac{d\phi}{dR}; \tag{4.58}$$

and thus the solution for the mean zonal pressure function becomes

$$f = (\beta + z)\left\{jR - \tfrac{1}{2}\phi_1^2 \int_{-R}^{0} \text{sn}^2(-\tfrac{1}{2}R\phi_2\sqrt{(1+2\gamma^2)}, k)\, dR\right\} - 3\gamma\phi(R) + \text{constant}. \tag{4.59}$$

In the case of zero momentum transfer ($\gamma = 0$), it was noted that $\partial f/\partial F$ maintains the same sign for all values of R in the range $-R_0 \leqslant R \leqslant 0$, but in (4.46) it will be observed that

$$\left(\frac{\partial f}{\partial R}\right)_{R=0} = (\beta + z)j + \tfrac{3}{2}\gamma\sqrt{(1+2\gamma^2)}\,\phi_1\phi_2, \tag{4.60}$$

$$\left(\frac{\partial f}{\partial R}\right)_{R=-R_0} = (\beta + z)j - \tfrac{3}{2}\gamma\sqrt{(1+2\gamma^2)}\,\phi_1\phi_2, \tag{4.61}$$

so that if γ is negative and sufficiently large (which ensures a transfer of westerly angular momentum towards $r = 0$) or if β is sufficiently small, then the possibility exists of a reversal in the direction of the mean zonal velocity. This suggests that the presence of easterlies embedded in a westerly zonal flow implies an angular momentum flow, as is well known to meteorologists. Using (4.48), we can say that if $\gamma < 0$ and

$$|\gamma \sqrt{(1+2\gamma^2)}\,\phi_1\phi_2| > \tfrac{2}{3}\beta, \tag{4.62}$$

then easterlies will appear near the outer (hotter) cylinder.

It is of some interest in this case to determine the meridional part of the vertical velocity which is given by (3.18). We have

$$\begin{aligned}\frac{r^4(\partial \overline{w}_1/\partial z)}{\epsilon^3 m^{-4}(1+z^2)\gamma} &= \frac{1}{2}\frac{d^2}{dR^2}(\phi^3) - \frac{d}{dR}(\phi^3) \\ &= 3\phi\left\{\tfrac{1}{2}\phi\frac{d^2\phi}{dR^2} + \left(\frac{d\phi}{dR}\right)^2 - \phi\frac{d\phi}{dR}\right\}.\end{aligned} \tag{4.63}$$

Near the boundary walls the right-hand side behaves like $3\phi(d\phi/dR)^2$; hence the mean vertical velocity will be upwards ($w_1 = 0$, $z = 0$). At the position of maximum ϕ, we have $d\phi/dR = 0$; hence the right-hand side is $\frac{3}{2}\phi^2 d^2\phi/dR^2$, which is necessarily negative. Hence, in the neighbourhood of the maximum value of ϕ, the mean vertical velocity is downwards. This velocity implies the existence of at least two cells, a direct cell near the outer cylinder and an indirect cell near the inner cylinder, and by considering the changes of sign of $\frac{1}{2}\phi\phi'' + \phi'^2 - \phi\phi'$ it may be verified that there are two and only two zeros of $\overline{w}_1$ between $R = 0$ and $R_1 = -R_0$, and hence there are only two cells in the meridional structure.

Case IV. The open dishpan problem. Non-existence of a continuous solution for ϕ

In this case we assume that the inner cylinder is removed and the liquid is enclosed by one cylinder of radius b. The boundary conditions to be satisfied by the function ϕ are now as follows. At the outer cylinder the normal velocity must vanish hence as before we must have $\phi(0) = 0$. At the central axis $r = 0$, or $R = -\infty$, since there is no source along the axis it is necessary that $\partial p/\partial\theta$ should vanish along the axis, hence $\phi(-\infty) = 0$. In addition, since u_θ must vanish at $r = 0$, it is necessary that $\phi'(-\infty) = 0$. Hence we must have

$$\phi(0) = 0, \quad \phi(-\infty) = \phi'(-\infty) = 0.$$

Since two derivatives vanish at $R = -\infty$, it follows from (4.38) that all the derivatives will vanish there, hence $\phi \equiv 0$. Thus, no continuous non-zero solution exists.

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